THE CHINESE UNIVERSITY OF HONG KONG

DEPARTMENT OF MATHEMATICS

MATH3070 (Second Term, 2017–2018)

Introduction to Topology

Exercise 4a Continuity

Remarks

Many of these exercises are adopted from the textbooks (Davis or Munkres). You are suggested to work more from the textbooks or other relevant books.

1. Do the exercises mentioned in lectures or in lecture notes.
2. Let $(X, \mathfrak{T}_X)$ and $(Y, \mathfrak{T}_Y)$ be metric spaces. Can a function $f\colon X \to Y$ be continuous if $\mathfrak{T}_Y$ is discrete?
3. Let $\mathbb{R}_{\ell\ell}$ be the real line with lower-limit topology (generated by $[a, b)$) and $\mathbb{R}$ be the standard real line. Give an example of continuous $f\colon \mathbb{R}_{\ell\ell} \to \mathbb{R}$ but it is not continuous when regarded as $\mathbb{R} \to \mathbb{R}$. Is there such an example for $\mathbb{R} \to \mathbb{R}_{\ell\ell}$?
4. Let $f\colon (X, \mathfrak{T}_X) \to (Y, \mathfrak{T}_Y)$ and $\mathcal{S}_Y$ be a subbase for $\mathfrak{T}_Y$. Is it true that f is continuous if and only if for every $S \in \mathcal{S}_Y$, $f^{-1}(S) \in \mathfrak{T}_X$?
5. Given a metric space (X, d), show that for each fixed $x_0 \in X$, the function

   $$x \mapsto d(x, x_0) : X \to \mathbb{R}$$

   is continuous.
6. Given a metric space (X, d) and a subset $A \subset X$, define $f\colon X \to [0, \infty)$ by $f(x) = \inf \{ d(x, a) : a \in A \}$. Show that f is a continuous function.
7. Let $f, g\colon X \to \mathbb{R}$ be continuous functions. Prove that the following sets are respectively open and closed, $\{ x \in X : f(x) < g(x) \}$, and $\{ x \in X : f(x) \leq g(x) \}$.

   This can be generalized if $\mathbb{R}$ is replaced with Y of ordered topology.
8. Let $(X, \mathfrak{T})$ be a topological space and $A \subset X$. The subspace (or induced or relative) topology on A is $\mathfrak{T}|_A = \{ G \cap A : G \in \mathfrak{T} \}$. Suppose $\mathfrak{T}_A$ is another topology on A. Find a necessary and sufficient condition for the inclusion map $\iota\colon (A, \mathfrak{T}_A) \to (X, \mathfrak{T})$ such that $\mathfrak{T}_A = \mathfrak{T}|_A$.
9. Let $(X, \mathfrak{T}_X)$ and $(Y, \mathfrak{T}_Y)$ be topological spaces. The (finite) product topology on $X \times Y$ is

   $$\mathfrak{T}_{X \times Y} = \{ U \times V : U \in \mathfrak{T}_X, V \in \mathfrak{T}_Y \} .$$

   Show that the projection mapping $\pi_X : X \times Y \to X$ is both open and continous.
10. Prove that the distance function $d\colon X \times X \to [0, \infty)$ is continuous if $X \times X$ is given the above product topology.

11. Refer to the above product topology on $X \times Y$, show that a function $f\colon\ Y \to X \times Y$ is continuous if and only if both $\pi_X \circ f\colon\ Y \to X$ and $\pi_Y \circ f\colon\ Y \to X$ are continuous.

12. Let $X \times X$ be given the product topology of X. Show that $D = \{ (x,x) : x \in X \}$ as a subspace of $X \times X$ is homeomorphic to X.

13. Given a metric space (X, d), define a function $\rho : X^2 \times X^2 \to [0, \infty)$ by

$$\rho((x_1, x_2), (y_1, y_2)) = \max \{ d(x_1, y_1), d(x_2, y_2) \} .$$

It is known that ρ is a metric on X^2. Refer to this metric ρ, prove that the distance function $d\colon\ X \times X \to [0, \infty)$ is continuous.